INDIAN STATISTICAL INSTITUTE, BANGALORE CENTRE
B.MATH - Third Year, 2012-13
Statistics - III, Semesteral Examination, December 3, 2012
Marks are shown in square brackets. Total Marks: 50

1. Let $n \geq 4$ and let $Z_i, 1 \leq i \leq n$ be independent $N(0, \sigma^2)$ random variables; let $0 < \alpha < 1$. Define $X_1 = Z_1$ and $X_{i+1} = \alpha X_i + \sqrt{1-\alpha^2} Z_{i+1}$ for $1 \leq i \leq n-1$. Let $\mathbf{X} = (X_1, \ldots, X_n)'$.
(a) Find the probability distribution of $\mathbf{X}$.
(b) Find the partial correlation coefficients $\rho_{12.3}$ and $\rho_{12.34}$ (between elements of $\mathbf{X}$). [10]

2. Consider the model $\mathbf{Y} = X\beta + \epsilon$, where $\epsilon \sim N_n(\mathbf{0}, \sigma^2 I_n)$ and $X_{n\times p}$ may not have full column rank but has $\mathbf{1}$ as its first column. Derive the joint distribution of $\bar{y}$ and the Residual Sum of Squares. [10]

3. Consider the problem of comparing $k \geq 2$ treatments. Suppose that under treatment i, the response $Y \sim N(\mu_i, \sigma^2)$, $1 \leq i \leq k$. If independent random samples of sizes $n_1, \ldots, n_k$, respectively, are available from groups of subjects who have undergone these treatments, describe the methodology for comparing the treatments. Show that the method reduces to a Student's t-test when $k = 2$. [10]

4. Consider the model:

$$y_{ijk} = \mu + \alpha_i + \tau_j + \delta_{ij} + \epsilon_{ijk},$$

$1 \leq i \leq I$, $1 \leq j \leq J$, $1 \leq k \leq K$, where ϵ_{ijk} are i.i.d. $N(0, \sigma^2)$ and with the usual constraints on the parameters for identifiability.
(a) Show that the least squares estimators of the parameters α_i, τ_j and δ_{ij} are also their maximum likelihood estimators.
(b) Find the maximum likelihood estimator of σ^2. Is it unbiased? [10]

5. Suppose $\mathbf{X} \sim N_p(\mu, \Sigma)$ where $\mathrm{Rank}(\Sigma) = r \leq p$ and let B be any symmetric matrix such that $B\mu = \mathbf{0}$. Show that $\mathbf{X}'B\mathbf{X}$ has a χ^2 distribution if and only if

$$\Sigma B \Sigma B \Sigma = \Sigma B \Sigma.$$

Find the degrees of freedom of such a χ^2 distribution. [10]